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k-Jacobsthal and k-Jacobsthal Lucas Numbers and their Associated Numbers

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Abstract: In this paper we define the associated k-jacobsthal numbers and associated k -jacobsthal lucas numbers and we give Diophantine triples for them.

1. INTRODUCTION:
The Fibonacci sequence and the lucas are the two shining stars in the vast array of integer sequences. They have fascinated both amateurs and professional mathematicians for centuries. Most recently Fibonacci, lucas, pell, pell-lucas, modified pell, jacobsthal, jacobsthal lucas sequences were generalized for any positive real number k. also the study of the k - Fibonacci, k - lucas, k - pell, k -pell-lucas, k - jacobsthal, k - jacobsthal lucas appeared . H.Salas has given several formulae for about k - fibonacci numbers and their associated numbers .

In this paper we have to give associated k -jacobsthal and associated k -jacobsthal lucas numbers Besides the usual jacobsthal numbers many kinds of generalization of these have been presented and well known jacobsthal sequence is defined as $j_0 = 0, j_1 = 1, j_n = j_{n-1} + 2\, j_{n-2}$, for $n \geq 2$where j_n denotes the n-jacobsthal numbers for any positive real number k, the k-jacobsthal sequence say $\{\{j_{k,n}\}_{n=1}^{\infty}$ is defined recurrently by

$$j_{k,n+1} = k\, j_{k,n} + 2\, j_{k,n-1} \text{ with initial conditions as} j_{k,0} = 0,\ j_{k,1} = 1$$

n	$j_{k,n}$
0	0
1	1
2	k
3	$k^2 + 2$
4	$k^3 + 4k$
5	$k^4 + 6k^2 + 4$
6	$k^5 + 8k^3 + 12k$
7	$k^6 + 10k^4 + 24k^2 + 8$
8	$k^7 + 12k^5 + 40k^3 + 32k$
9	$k^8 + 14\,k^6 + 60k^4 + 80k^2 + 16$
10	$k^9 + 16\,k^7 + 84k^5 + 16k^3 + 80k$

Particular cases of the previous definitions are if k=1, the classical jacobsthal sequence obtained $j_0 = 0, j_1 = 1, j_{n+1} = j_n + 2\, j_{n-1}$ for $n \geq 1$, $\{j_{k,n}\}_{n=0}^{\infty} = \{0,1,1,3,5,11 \ldots.\}$

The Associated $k -$ jacobsthal numbers

We define the sequence $\{A_{k,n}\}_{n=0}^{\infty}$ associated to $\{j_{k,n}\}_{n=0}^{\infty}$ as $A_{k,0} = 1, A_{k,1} = 1$,
$A_{k,n} = kj_{k,n} + 2j_{k,n-1}\ for\ n \geq 1,2 \ldots.$

Observe that for $n = 1,2,3$ ... the expression $A_{k,n}$ is the sum of the two consecutive $k -$jacobsthal numbers $j_{k,n}$ and its predecessor $j_{k,n-1}$. The numbers of the sequence $\{A_{k,n}\}_{n=0}^{\infty}$ will be called associated k-jacobsthal numbers. An equlent definition for the sequence $\{A_{k,n}\}_{n=0}^{\infty}$ is

$$A_{k,n} = \begin{cases} 1, & n = 0 \\ 1, & n = 1 \\ (k+2)j_{k,n-1} + 2\, j_{k,n-2} & n \geq 2 \end{cases}$$

Observe that

$$\begin{aligned} A_{k,n} &= j_{k,n} + 2j_{k,n-1} \\ &= (kj_{k,n-1} + 2j_{k,n-2}) + 2\,(kj_{k,n-2} + 2j_{k,n-3}) \\ &= (k(j_{k,n-1} + 2j_{k,n-2}) \;+ 2\,(j_{k,n-2} + 2j_{k,n-3}) \\ &= kA_{k,n-1} + 2A_{k,n-2} \end{aligned}$$

This allows to define recursively the sequence of associated k-jacobsthal numbers as follows

$$A_{k,n} = \begin{cases} 1, & n = 0 \\ 1, & n = 1 \\ kA_{k,n-1} + 2A_{k,n-2} & n \geq 2 \end{cases} \tag{1}$$

following table shows some of the associated k-jacobsthal numbers

n	$A_{k,n}$
0	1
1	1
2	k+2
3	$k^2 + 2k + 2$
4	$k^3 + 2\,k^2 + 4k + 4$
5	$k^4 + 2k^3 + 6k^2 + 8k + 4$
6	$k^5 + 2k^4 + 8k^3 + 12k^2 + 12k + 8$
7	$k^6 + 2k^5 \;+ 10k^4 + 16k^3 + 24k^2 + 8$
8	$k^7 + 2k^6 + 12k^5 + 20k^4 + 40k^3 + 48\,k^2 + 32k + 16$
9	$k^8 + 2k^7 + 14\,k^6 + 24k^5 + 60k^4 + 80k^3 + 80k^2 + 16k + 16$
10	$k^9 + 2k^8 + 16\,k^7 + 24k^6 + 84k^5 + 120k^4 + 160k^3 + 112k^2 + 80k + 32k$

$If\ k = 1$ the associated k-jacobsthal sequence are $A_{k,0}$ = 1, $A_{k,1}$ =1 and {1,1,3,5,11,21….}
$If\ k = 2$ the associated k-jacobsthal sequence {1,1,4,12……}

Remark 1 :

We can obtain Diophantine triples for $D(k+1)$ for associated k-jacobsthal numbers

We search for three distinct integers a, b, c which are k-jacobsthal numbers such that the product of any two from the set added with $2^{2n-1}(k+1)$ is a perfect square

$$A_{k,(2n-1)}c + 2^{2n-1}(k+1) \quad = \quad \alpha^2 \tag{2}$$

$$A_{k,(2n+1)}c + 2^{2n-1}(k+1) \quad = \quad \beta^2 \tag{3}$$

Form (2)and(3)

$$A_{k,(2n+1)}\,\alpha^2 - A_{k,(2n-1)}\,\beta^2 = 2^{2n-1}(k+1)\,[A_{k,(2n+1)} - A_{k,(2n-1)}\,] \tag{4}$$

$$\alpha = X + A_{k,(2n-1)}\,T \qquad \beta = X + A_{k,(2n+1)}T$$

in (4), we get

$$X = A_{k,2n} \text{ and } \alpha = A_{k,2n} + A_{k,2n-1}$$

From (2)

c = $2A_{k,2n} + A_{k,2n-1} + A_{k,2n+1}$

Simillarly we can find

d =$4A_{k,2n} + A_{k,2n-1} + 4A_{k,2n+1}$

some numerical values

n	k	a	b	c
1	1	1	5	12
2	1	5	21	48
3	1	21	85	192

Remark 2 :

Simillarly we have to obtain $D(k+1)$ for associated k-jacobsthal numbers. The product of any two numbers from the set minus $2^{2n}(k+1)$ is a perfect square.
$a = A_{k,2n}$ $b = A_{k,2(n+1)}$ thus the triple (a, b, c) for associated k −jacobsthal numbers

$$A_{k,2n}A_{k,2(n+1)} - 2^{2n}(k+1) = X^2$$

The results will be

$$(A_{k,2n}, A_{k,2(n+1)}, \left(A_{k,2(n+1)} + A_{k,2n} + A_{k,2n+1}\right), (4A_{k,2n+1} + 4A_{k,2(n+1)} + A_{k,2n}))$$

n	a	b	c	d
1	$A_{k,2}$	$A_{k,4}$	$A_{k,4} + A_{k,2} + 2A_{k,3}$	$4A_{k,4} + A_{k,2} + 4A_{k,3}$
2	$A_{k,4}$	$A_{k,6}$	$A_{k,6} + A_{k,4} + 2A_{k,5}$	$4A_{k,6} + A_{k,4} + 4A_{k,5}$
3	$A_{k,6}$	$A_{k,8}$	$A_{k,8} + A_{k,6} + 2A_{k,7}$	$4A_{k,8} + A_{k,6} + 4A_{k,7}$

some numerical values

n	k	a	b	c
1	1	1	5	12
2	1	5	21	48
3	1	21	85	192

Section II

In this section we have to fine associated k −jacobsthal lucas numbers. For any positive real numbers k, the k-jacobsthal $\{\hat{j}_{k,n}\}_{n=0}^{\infty}$ sequences are defined recurrently by

$$\hat{j}_{k,n} = k\hat{j}_{k,n-1} + 2\hat{j}_{k,n-2} \quad , \qquad \hat{j}_{k,0} = 2, \hat{j}_{k,1} = k,\ n \geq 2$$

$\boldsymbol{n}$	$\boldsymbol{\hat{j}_{k,n}}$
0	2
1	k
2	$k^2 + 4$
3	$k^3 + 6k$
4	$k^4 + 8k^2 + 8$
5	$k^5 + 10k^3 + 20k$
6	$k^6 + 12k^4 + 36k^2 + 16$
7	$k^7 + 14k^5 + 56k^3 + 56k$
8	$k^8 + 16\,k^6 + 80k^4 + 128k^2 + 32$
9	$k^9 + 18\,k^7 + 108k^5 + 140k^3 + 144k$
10	$k^{10} +\ 20\,k^8 + 140k^6 + 300k^4 + 272k^2 +\ 64$

Particular cases of the previous definitions are

- if k=1 and $\hat{j}_{k,0} = 2, \hat{j}_{k,1} = 1$, the classic jacobsthal lucas sequence is obtained

Associated *k*- jacobsthal lucas numbers

We define the sequence $\{A_{k,n}\}^{\infty}_{n=0}$ associated to $\{\hat{j}_{k,n}\}^{\infty}_{n=0}$ as

$$A_{k,n} = \begin{cases} 2, & n = 0 \\ k+4, & n = 1 \\ k\hat{j}_{k,n-1} + 2\hat{j}_{k,n-2} & n \geq 2 \end{cases}$$

Observe that

$$\begin{aligned} A_{k,n} &= \hat{j}_{k,n} + 2\,\hat{j}_{k,n-1} \\ &= (k\hat{j}_{k,n-1} + 2\hat{j}_{k,n-2}) + 2(k\hat{j}_{k,n-2} + 2\hat{j}_{k,n-3}) \\ &= k(\hat{j}_{k,n-1} + \hat{j}_{k,n-2}) + 2\,(\hat{j}_{k,n-2} + 2\hat{j}_{k,n-3}) \\ &= k\,A_{k,n-1} + 2\,A_{k,n-2} \end{aligned}$$

This allows to define recursively the sequence of associated k-jacobsthal lucas numbers as follows

$$A_{k,n} = \begin{cases} 2, & n = 0 \\ k+4, & n = 1 \\ k\,A_{k,n-1} + 2\,A_{k,n-2} & n \geq 2 \end{cases}$$

following table shows some of the associated k-jacobsthal lucas numbers

$\boldsymbol{n}$	$\boldsymbol{A_{k,n}}$
0	2
1	k+4
2	$k^2 + 4k + 4$
3	$k^3 + 4\,k^2 + 6k + 8$
4	$k^4 + 4k^3 + 8k^2 + 16k + 8$
5	$k^5 + 4k^4 + 10k^3 + 24k^2 + 20k + 16$
6	$k^6 + 4k^5 + 12k^4 + 32k^3 + 36k^2 + 48k + 16$
7	$k^7 + 4k^6 + 14k^5 + 40k^4 + 56k^3 + 96\,k^2 + 56k + 16$
8	$k^8 + 4k^7 + 16\,k^6 + 48k^5 + 80k^4 + 160k^3 + 128\,k^2 + 128k + 32$
9	$k^9 + 4k^8 + 18\,k^7 + 56\,k^6 + 108k^5 + 240k^4 + 240k^3 + 320\,k^2 + 144k + 32$
10	$k^{10} + k^9 + 4k^8 + 18\,k^7 + 56\,k^6 + 108k^5 + 240k^4 + 240k^3 + 320\,k^2 + 144k + 32$

If $k = 2$ the sequence are { 2,6,16….}

Remark 3:

We can obtain Diophantine triples for D(k^2+8) for associated k-jacobsthal lucas numbers

$a = A_{k,2n-1}$ $\quad b = A_{k,2n+1}$

ab +2^{2n-1}(k^2+8) is a perfect square, Repeating the procedure as mentioned in section I, we can take the values of c and d

$C = A_{k,2n+1} + A_{k,2n-1} + 2A_{k,2n}$ $\qquad d = 4A_{k,2n+1} + A_{k,2n-1} + 4A_{k,2n}$

n	a	b	c	d
1	$A_{k,1}$	$A_{k,3}$	$A_{k,1} + 2A_{k,2} + A_{k,3}$	$A_{k,1} + 4A_{k,2} + 4A_{k,3}$
2	$A_{k,3}$	$A_{k,5}$	$A_{k,3} + 2A_{k,4} + A_{k,5}$	$A_{k,3} + 4A_{k,4} + 4A_{k,5}$
3	$A_{k,5}$	$A_{k,7}$	$A_{k,5} + 2A_{k,6} + A_{k,7}$	$A_{k,5} + 4A_{k,6} + 4A_{k,7}$

some numerical values

n	k	a	b	c	d
1	1	5	19	42	117
2	1	19	75	168	467
3	1	75	283	656	1803

Remark 4 :

Simillarly the same procedure may be applied for associated k-jacobsthal numbers. $a = A_{k,2n}$ $b = A_{k,2(n+1)}$

$ab - 2^{2n}(k^2 + 8)$ is a perfect square. Repeating the same procedure in section I, we get the values c and d. $(A_{k,2n}, A_{k,2(n+1)}, c, d)$

$C = A_{k,2(n+1)} + A_{k,2n} + 2A_{k,2n+1}$ $\quad\quad d = 4A_{k,2(n+1)} + A_{k,2n} + 4A_{k,2n+1}$

Some numerical values

n	k	a	b	c	d
1	1	9	37	54	263
2	1	37	149	336	933
3	1	149	597	1312	3669

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